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ON NONCONSERVATIVE PERIODIC SYSTEMS CLOSE TO TWO-DIMENSIONAL HAMILTONIAN*

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Time-periodic perturbations of two-dimensional nonlinear Hamiltonian systems are investigated. Methods of the analysis of truncated systems are used to establish, in the neighborhoods of the individual closed energy levels, the possible types of qualitative behavior of the solutions, and their classification is given. The most interesting resonance case is studied in detail, and the existence of essentially new two-frequency resonance modes is indicated. The passage to the nonresonant case when the exact resonance is detuned, is studied.

1. Let us consider the system

$$x^{\cdot} = \partial H(x, y)/\partial y + \varepsilon g(x, y, \nu t), \quad y^{\cdot} = -\partial H(x, y)/\partial x + \varepsilon f(x, y, \nu t) \tag{1.1}$$

Here H, g, f are functions continuous and 2π-periodic in $\varphi = \nu t$, and sufficiently smooth in x and y in some region D, ν is a parameter and ε is a small positive parameter. Let us investigate the behavior of solutions of the system (1.1) on the sets $D_j \times S^1$ where D_j $(1 \leqslant j \leqslant k)$ denote the compact invariant regions filled with closed trajectories of the unperturbed system and not containing the small neighborhoods of the centers, the separatrix contours, the parabolic trajectories, nor the "infinity". The solutions of (1.1) in $D_j \times S^1$ are characterized by the possibility of separating the variables into the "rapid" and "slow".

Let $D_j \times S^1$ be one of the regions in which $D_j = \{(x, y): h_{j1} < H(x, y) < h_{j2}, h_{j1}, h_{j2} = \text{const}\}$. In what follows, we shall omit the subscript j. Using a canonical transformation related to the passage to the action I-angle θ variables, we can transform the system (1.1) in this region to the form

$$\begin{aligned} I^{\cdot} &= \varepsilon(fx_\theta' - gy_\theta') \equiv \varepsilon F_1(I, \theta, \varphi) \\ \theta^{\cdot} &= \omega(I) + \varepsilon(-fx_I' + gy_I') \equiv \omega(I) + \varepsilon F_2(I, \theta, \varphi) \end{aligned} \tag{1.2}$$

The functions $F_{1,2}(I, \theta, \varphi)$ are 2π-periodic in $\varphi = \nu t$ and θ. The phase space of the system (1.2) is a direct product $K \times T^2$ where K denotes the interval $(I_1 = I(h_1), I_2 = I(h_2))$ while $T^2 = S^1 \times S^1$ is a two-dimensional torus. In addition to (1.2), we consider an autonomous system defined on the ring $K \times S^1$

$$I^{\cdot} = \varepsilon B_0(I), \quad \theta^{\cdot} = \omega(I) + \varepsilon Q(I) \tag{1.3}$$

obtained from the system

$$x^{\cdot} = \partial H/\partial y + \varepsilon \bar{g}(x, y), \quad y^{\cdot} = -\partial H/\partial x + \varepsilon \bar{f}(x, y) \tag{1.4}$$

after passing to the variables I and θ and averaging over θ. Here

$$B_0(I) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_1(I, \theta, \varphi)\, d\varphi\, d\theta$$

$$Q(I) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_2(I, \theta, \varphi)\, d\varphi\, d\theta$$

$$\bar{g}(x, y) = \frac{1}{2\pi}\int_0^{2\pi} g(x, y, \varphi)\, d\varphi, \quad \bar{f}(x, y) = \frac{1}{2\pi}\int_0^{2\pi} f(x, y, \varphi)\, d\varphi$$

We assume that the generating Poincaré–Pontriagin equation

$$B_0(I) = 0 \tag{1.5}$$

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.385-394,1983 has at most a finite number of real roots which are all simple. This implies that

$$B_1(I_*) \equiv \frac{dB_0(I_*)}{dI} = \frac{1}{2\pi}\int_0^{2\pi} (\bar{g}_x' + \bar{f}_y')\big|_{\substack{x=x(I_*,\,\theta)\\ y=y(I_*,\,\theta)}} d\theta \neq 0 \tag{1.6}$$

where I_* is a root of the equation (1.5). From (1.5), (1.6) and the small parameter method it follows that for sufficiently small $\varepsilon \neq 0$ the limit cycles of the system (1.4) are coarse. The problem of investigating the system (1.2) is equivalent to the study of the mapping T_ε of the plane $\varphi = 0$ onto the plane $\varphi = 2\pi$:

$$\bar{I} = I + \varepsilon\frac{2\pi}{\nu} A_0(I,\theta) + O(\varepsilon^2), \quad \bar{\theta} = \theta + \frac{2\pi}{\nu}\omega(\overline{I}) + \varepsilon\frac{2\pi}{\nu} Q_0(I,\theta) + O(\varepsilon^2)$$

$$\left(A_0(I,\theta) = \frac{1}{2\pi}\int_0^{2\pi} F_1\left(I, \theta + \frac{\omega\varphi}{\nu}, \varphi\right) d\varphi\right.$$

$$Q_0(I,\theta) = \frac{1}{2\pi}\int_0^{2\pi} F_2\left(I, \theta + \frac{\omega\varphi}{\nu}, \varphi\right) d\varphi$$

When $\varepsilon = 0$, all closed curves $I = \text{const}$ of the mapping T_0 are invariant, with the Poincaré rotation number equal to $2\pi\omega/\nu$. If this number of rotation is commensurate with 2π, i.e. if $\omega = (q/p)\nu$, q, p are integers, then the level is called resonant and denoted by $I = I_{pq}$, otherwise the level $I = \text{const}$ is called nonresonant. Consider the mapping T_ε. The small parameter method implies that if the generating equation

$$\int_0^{2\pi p} F_1\left(I_{pq}, \theta + \frac{q\varphi}{p}, \varphi\right) d\varphi = 0$$

has a simple root $\theta = \theta_*$, then periodic points are generated from the level $I = I_{pq}$. The initial system (1.1) has a stable solution $2\pi p$-periodic in φ corresponding to the values $I = I_{pq}, \theta = \theta_*$, provided that

$$\sigma(I_{pq},\theta_*) = A_{0I}'(I_{pq},\theta_*) + Q_{0\theta}'(I_{pq},\theta_*) < 0$$
$$\omega'(I_{pq})\, A_{0\theta}'(I_{pq},\theta_*) < 0$$

If, on the other hand, $\omega'(I_{pq})\, A_{0\theta}'(I_{pq},\theta_*) > 0$, then the values $I = I_{pq}$, $\theta = \theta_*$ have a periodic solution of the saddle type, with the same period. We note that

$$\sigma(I_{pq},\theta) = \frac{1}{2\pi p}\int_0^{2\pi p} (g_x' + f_y')\big|_{\substack{x=x(I_{pq},\,\theta+q\varphi/p)\\ y=y(I_{pq},\,\theta+q\varphi/p)}} d\varphi \tag{1.7}$$

and the mean value of σ is equal to B_1. If the level $I = I_*$ is nonresonant and I_* is a root of the equation (1.5), then under the known conditions a closed invariant curve of the mapping T_ε /1/, stable at $\varepsilon B_1(I_*) < 0$, exists near this level.

In the region $K \times T^2$ the system (1.1) usually does not admit an unambiguous study by means of the method of averaging. For this reason the investigation of the system (1.2) involves the analysis, in a certain ε-dependent neighborhood U of every individual level $I = \text{const}$, followed by description of the behavior of the solutions in $K \times T^2$. A similar approach was used in /2,3/ as investigation of Duffing type equations and also /4/ in the first conservative approximation to the equation $\theta'' = \varepsilon F(\theta', \theta, \nu t)$ where F is a function 2π-periodic in θ and a function $\varphi = \nu t$.

Below we study the structure of solutions in small ε-dependent neighborhoods of the individual levels $I = \text{const}$. We concentrate our attention on describing the nonlinear resonances and their bifurcations. The condition $B_1(I_*) \neq 0$ enables us to assert that the second order approximation of the method of averaging is nonconservative. We note that if the function $\sigma(\theta, I_{pq})$ is sign definite and equation (1.5) has no real roots, then the study of such resonances in the conceptual plane is analogous to that carried out in /3/ but in the presence of real roots in the equation (1.5), then is in /2/. If, on the other hand, the function σ has a variable signature, then essentially new two-frequency modes may exist in the system (1.1). We also study the degenerate resonances $I = I_v$

$$d^r\omega(I_v)\,/\,dI^r = 0, \quad r = 1, 2, \ldots, m \tag{1.8}$$
$$d^{m+1}\omega(I_v)\,/\,dI^{m+1} \neq 0$$

(Similar resonances can exist in cells the boundaries of which include two separatrix contours). We use the method of averaging /5,6/, as well as those of the qualitative theory and the theory of bifurcation of dynamic systems on a plane /7-9/.

2. The problem of reducing the system (1.2) in the neighborhoods of the individual levels to a more convenient form, can be split into four cases: 1°. In the neighborhoods $U_{\sqrt{\varepsilon}} = \{(I, \theta) : I_{pq} - c\sqrt{\varepsilon} < I < I_{pq} + c\sqrt{\varepsilon},\ 0 \leqslant \theta < 2\pi\}$ of the order of $\sqrt{\varepsilon}$ in I of the resonant nondegenerate levels $I = I_{pq}$; 2°. In the neighborhoods of the order of $\varepsilon^{1/(m+2)}$ in I of the resonant degenerate levels; 3°. In the neighborhoods of the order of $\sqrt{\varepsilon}$ in I of the nonresonant nondegenerate levels; 4°. In the neighborhoods of the order of $\varepsilon^{1/(m+2)}$ in I of the nonresonant degenerate levels.

1°. System (1.2) is reduced to the form /10/

$$u^{\cdot} = \mu A_0(\psi, I_{pq}) + \mu^2\sigma(\psi, I_{pq})\, u + O(\mu^3) \qquad (2.1)$$
$$\psi^{\cdot} = \mu b u + \mu^2 b_1 u^2 + O(\mu^3)$$

where

$$A_0(\psi, I_{pq}) \equiv A_{0*}(\psi, I_{pq}) + B_0(I_{pq}) = \frac{1}{2\pi p}\int_0^{2\pi p} F_1\left(I_{pq}, \psi + \frac{q\varphi}{p}, \varphi\right) d\varphi \qquad (2.2)$$

is a function $2\pi/p$ -periodic in ψ while $\sigma(\psi, I_{pq})$ is given by (1.7). Terms of the order of μ^3 in (2.1) depend on u, ψ and φ and are 2π - and $2\pi p$ -periodic in ψ and φ respectively. According to /3,10/ the passage from (1.2) to (2.1) can be carried out in three stages: 1) $(I, \theta) \to (h, \Phi)$, 2) $(h, \Phi) \to (W, \psi)$, 3) $(W, \varphi) \to (u, \psi)$, and here, in contrast to /3,10/, in the first stage we carry out the following identity substitution on the torus $\{(\theta, \varphi) \bmod 2\pi\}$:

$$I = I_{pq} + \mu h, \quad \theta = \Phi + q\varphi/p$$

2°. The substitution

$$I = I_{pq} + \varepsilon^s h, \quad \theta = \Phi + q\varphi/p, \quad s = 1/(m+2)$$

reduces the system (1.2) to the form

$$h^{\cdot} = \varepsilon^{1-s} F_1(I_{pq}, \Phi + q\varphi/p, \varphi) + \varepsilon F_{1I}'(I_{pq}, \Phi + q\varphi/p, \varphi)\, h + O(\varepsilon^{1+s})$$

$$\Phi^{\cdot} = \varepsilon^{1-s} b_m h^{m+1} + \varepsilon\,(b_{m+1} h^{m+2} + F_2(I_{pq}, \Phi + q\varphi/p, \varphi)) + O(\varepsilon^{1+s})$$

Further, following 1° we arrive at the system

$$u^{\cdot} = \varepsilon^{1-s}(A_{0*}(\psi, I_{pq}) + B_0(I_{pq})) + \varepsilon\,(P_{0*}(\psi, I_{pq}) + B_1(I_{pq}))\, u + O(\varepsilon^{1+s}) \qquad (2.3)$$
$$\psi^{\cdot} = \varepsilon^{1-s} b_m u^{m+1} + \varepsilon\,(b_{m+1} u^{m+2} + Q_0(\psi, I_{pq})) + O(\varepsilon^{1+s})$$

The terms $O(\varepsilon^{1+s})$ depend on u, ψ, φ; $b_m = d^{m+1}\omega(I_{pq})/dI^{m+1}(m+1)!$, $b_0 = b$; B_1, P_{0*}, Q_0 satisfy the condition $B_1 + P_{0*} + dQ_0/d\psi \equiv \sigma$.

3°. Let the level $I = I_*$ be nondegenerate and nonresonant. Carrying out in (1.2) the substitution $I = I_* + \mu h$, we arrive at the system

$$h^{\cdot} = \mu F_1(I_*, \theta, \varphi) + O(\mu^2), \quad \theta^{\cdot} = \omega_* + \mu b h + O(\mu^2), \quad \omega_* = \omega(I_*) \qquad (2.4)$$

Expanding the function F_1 into a double Fourier series (N is a given number)

$$F_1(I_*, \theta, \varphi) = \sum_{k,\, m=-N}^{N} F_{km}(I_*)\, e^{i(k\theta + m\varphi)} + R_N(I_*, \theta, \varphi) \qquad (2.5)$$

and passing in (2.4) to the variable u according to the formula

$$h = u - i\mu \sum_{\substack{k,\, m=-N \\ k^2+m^2\neq 0}}^{N} \frac{F_{km}(I_*)}{m\nu + k\omega_*}\, e^{i(k\theta + m\varphi)} \qquad (2.6)$$

we arrive at the system

$$u^{\cdot} = \mu\,(B_0(I_*) + R_N(I_*, \theta, \varphi)) + O(\mu^2), \quad \theta^{\cdot} = \omega_* + \mu b u + O(\mu^2) \qquad (2.7)$$

We know that the substitution (2.6) with $N = \infty$ is usually divergent, and we shall call the case of divergent substitution irreducible. The cases of convergence (e.g. if the ratio $\omega_* \nu$ is poorly approximated, in the accepted sense of the term, by rational numbers, or if the perturbation can be represented in the form of trigonometric polynomials in φ /1/), shall be called reducible.

4°. Lastly, let the level $I = I_*$ be degenerate and nonresonant. Then the system (1.2) reduces to the form

$$u^{\cdot} = \varepsilon^{1-s}(B_0(I_*) + R_N(I_*, \theta, \varphi)) + O(\varepsilon) \quad \theta^{\cdot} = \omega_* + \varepsilon^{1-s} b_m u^{m+1} + O(\varepsilon) \tag{2.8}$$

Subsequent investigation depends, essentially, on the following criteria:

1) The level $I = \text{const}$ is resonant or nonresonant; 2) the autonomous system (1.4) has or has not a limit cycle in the neighborhood of the level $I = \text{const}$ under consideration; 3) the function $\sigma(\psi)$ is or is not sign definite; 4) the level $I = \text{const}$ is degenerate or nondegenerate.

3. Let us turn our attention to the qualitative analysis of the behavior of solutions of the system (1.2) in the neighborhoods of the individual levels $I = \text{const}$, basing this on the analysis of the truncated systems obtained from (2.1), (2.3), (2.7), (2.8) by discarding the nonautonomous terms. We begin with the nondegenerate levels, denoting by I_0 the root of (1.5): $B_0(I_0) = 0$.

Case 1. $I = I_{pq} \neq I_0$. The case can be suitable split into two subcases.

1a. The equation

$$A_{0*}(\psi) + B_0 = 0 \tag{3.1}$$

has no real roots. In accordance with (2.1), in the U_μ neighborhood of such levels the qualitative behavior of solutions of the system (1.1) is analogous to the behavior of solutions of the autonomous system (1.4). Such resonant levels are naturally called penetrable.

1b. Equation (3.1) has real roots $\psi = \psi_j$. In this case the initial system (1.1) has periodic solutions, roughly half of which are of saddle type, and the other half are asymptotically stable when $\varepsilon\sigma(\psi_j) < 0$.

Let us consider a truncated system obtained from (2.1) by discarding terms of order $O(\mu^3)$

$$u^{\cdot} = \mu(A_{0*}(\psi) + B_0) + \mu^2\sigma(\psi)u, \quad \psi^{\cdot} = \mu bu + \mu^2 b_1 u^2 \tag{3.2}$$

The system (3.2) is defined on the cylinder $\{\psi \bmod 2\pi, u\}$. However, since the smallest period of the function $A_{0*}(\psi), \sigma(\psi)$ is equal to $2\pi/p$, it is sufficient to consider the behavior of solutions of the system in the strip $\{\psi \bmod (2\pi/p), u\}$.

Assertion 1. If the function $\sigma(\psi)$ is sign definite, then the truncated system (3.2) has no limit cycles.

Indeed, the absence of limit cycles not enveloping the phase cylinder follows from the Bendickson criterion /7/, and of the cycles enveloping the phase cylinder from the condition $B_0 \neq 0$. When the fraction $\sigma(\psi)$ has variable sign and $B_0 \neq 0$, the system (3.2) can have only the limit cycles not enveloping the phase cylinder. Therefore, irrespective of whether $\sigma(\psi)$ is, or is not sign definite, regions exist in the neighborhood U_μ of the resonant levels in question, for the initial conditions, from which the phase point leaves U_μ after a finite time, and regions in which the phase point remains in U_μ at all $t \to +\infty$ or $t \to -\infty$. We shall call such resonant levels partially penetrable. From (1.7) it follows that $\sigma(\psi)$ can be a sign variable function for the parametric systems for which the perturbation contains the term $a_{mn}(\nu t)x^m y^m$, $m, n \neq 0$.

Case 2. $I = I_{pq} = I_0$. In accordance with the assumption $B_0(I_0) = 0$, $B_1(I_0) \neq 0$, and the autonomous system (1.4) has a limit cycle in the neighborhood of the level in question. In this case the truncated system (3.2) assumes the form

$$u^{\cdot} = \mu A_{0*}(\psi) + \mu^2\sigma(\psi)u, \quad \psi^{\cdot} = \mu bu + \mu^2 b_1 u^2 \tag{3.3}$$

and is close to the Hamiltonian system

$$u^{\cdot} = \mu A_{0*}(\psi), \quad \psi^{\cdot} = \mu bu \tag{3.4}$$

Using the first integral $bu^2/2 - \int A_{0*}(\psi)\,d\psi = \text{const}$ we can establish the topological structure of the behavior of the solutions of the system (3.4). When terms of order μ^2 are taken into account, limit cycles can exist in (3.3) enveloping the phase cylinder $\{\psi \bmod 2\pi, u\}$. Since the "infinity" ($|u| = \infty$) is unstable when $\varepsilon B_1 < 0$ and stable when $\varepsilon B_1 > 0$, a strip $|u| \leqslant u_0$ can be separated on the phase cylinder into which all trajectories will converge from the outside when $\varepsilon B_1 < 0$, and from which all trajectories will emerge when $\varepsilon B_1 > 0$. Such resonant levels shall be called impenetrable.

When $\varepsilon B_1(I_{pq}) < 0$, a ring exists in the initial system in the neighborhood U_μ of the level $I = I_{pq}$ is question, into which all trajectories converge as $t \to \infty$. Stable and unstable oscillation synchronisation modes $2\pi p$-periodic in φ exist within the ring itself. When the function $\sigma(\psi)$ has sign-changing, two types of two-frequency modes are also possible: a) the nodes corresponding to the limit cycles of the system (3.3) not enveloping the phase cylinder, and b) those corresponding to the limit cycles of (3.3) enveloping the phase cylinder. In the case when $\sigma(\psi)$ has constant sign, it can be shown (see Sect.4) that no-frequency modes referred to above, exist.

Case 3. The level $I = I_*$ is nonresonant and $I_* \neq I_0$. In this case the number N in (2.7) is chosen so that $\max_{\theta,\varphi}|R_N| < |B_0|/2$. This leads us to a situation analogous to the case 1a, i.e. such levels are penetrable.

Case 4. The level $I = I_*$ is nonresonant and $I_* = I_0$. Here we have $B_0 = 0$ and in the reducible case the initial system has a two-dimensional invariant torus /1/, and the mapping T_ε has a closed smooth invariant curve. When $\varepsilon B_1(I_*) < 0$, the torus is asymptotically stable. In the irreducible case a ring can be shown to exist in the neighborhood of the level $I = I_*$ in question (with a boundary depending, generally, on θ), into which the trajectories of the mapping T_ε arrive when $\varepsilon B_1 < 0$.

Next we turn our attention to the degenerate cases, without concerning ourselves with the problem of existence of limit cycles in the truncated systems.

Case 5. $I = I_{pq} \neq I_0$ and condition (1.8) with $m \geqslant 1$ holds. According to (2.3) the behavior of the solutions in the neighborhood U_{ε^s} of the level $I = I_{pq}$ is described with the accuracy up to terms of order ε^{1+s}, $s = 1/(m+2)$ by the following truncated system:

$$
\begin{aligned}
u^\cdot &= \varepsilon^{1-s}(A_{0*}(\psi) + B_0) + \varepsilon(P_{0*}(\psi) + B_1)\,u \\
\psi^\cdot &= \varepsilon^{1-s} b_m u^{m+1} + \varepsilon(b_{m+1}\,u^{m+2} + Q_0(\psi)),\ m \geqslant 1
\end{aligned}
\tag{3.5}
$$

5a. If the equation $A_{0*}(\psi) + B_0 = 0$ has no real roots, then the level $I = I_{pq}$ is penetrable.

5b. Let the equation $A_{0*}(\psi) + B_0 = 0$ have real roots $\psi = \psi_j$. Consider the conservative system

$$
u^\cdot = \varepsilon^{1-s}(A_{0*}(\psi) + B_0),\quad \psi^\cdot = \varepsilon^{1-s} b_m u^{m+1} \tag{3.6}
$$

The "energy" integral of this system

$$
b_m u^{m+2}/(m+2) - \int A_{0*}(\psi)\, d\psi - B_0\psi = \text{const}
$$

enables us to construct the phase patterns. We note that the system (3.6) has only the fine states of equilibrium, with two zero roots of the characteristic equation. The phase pattern depends essentially on whether m is even or odd. When m is odd, we have the degenerate states of equilibrium of the "impenetrable crumb" type, while with m even we have topological saddles and centers /7/.

Let us consider the system (3.5). The coordinate u_1 of the state of equilibrium is found from the equation

$$
b_m u^{m+1} + \varepsilon^s Q_0(\psi_j) - \varepsilon^{2s}[Q_0'(\psi_j) \times (P_{0*}(\psi_j) + B_1)/A_0'(\psi_j)]\,u + \varepsilon^s u^{m+2} b_{m+2} = O(\varepsilon^{3s}) \tag{3.7}
$$

and coordinate ψ coincides, with the accuracy of up to terms of order $\varepsilon^{1/(m+1)}$, with ψ_j. The roots of characteristic equation are equal to

$$
\begin{aligned}
&\varepsilon\sigma(\psi_j)/2 \pm (\varepsilon^2\sigma^2(\psi_j)/4 - \Delta)^{1/2} \\
&\Delta = O(\varepsilon^2) - \varepsilon^{2(m+1)/(m+2)} A_{0*}'(\psi_j) b_m (m+1)\, u_1^{m}
\end{aligned}
$$

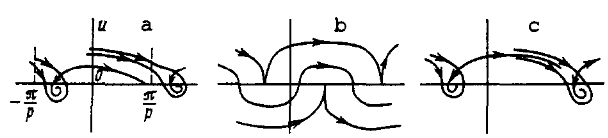

Fig.1

In accordance with (3.7), the second term in the expression for Δ is of the order of ε^λ where $\lambda = [2(m+1)^2 + m]/(m+1)(m+2)$. Since $3/2 \leqslant \lambda < 2$, it follows that the term $-\Delta$ is dominant in the discriminant $\varepsilon^2\sigma^2(\psi_j)/4 - \Delta$. Consequently, when $A_{0*}'(\psi_j) \neq 0$, $\sigma(\psi_j) \neq 0$, the equilibrium states will be coarse foci and saddles. When m is odd and conditions that $Q_0(\psi)$ has opposite signs at the neighboring zeros $A_0(\psi)$ holds, the trajectories of the system (3.5) may behave in the manner shown in Fig.1a. When m is even, the qualitative behavior of the trajectories of (3.5) is basically the same as in the nondegenerate case. It follows that in the present case the level $I = I_{pq}$ is partly penetrable.

Case 6. $I = I_{pq} = I_0$ and condition (1.8) holds. In this case $B_0(I_{pq}) = 0$, $B_1(I_{pq}) \neq 0$. Possible behavior of the trajectories of the system (3.6) with $B_0 = 0$ is shown in Fig.1b for odd m. Fig.1c shows the corresponding phase pattern of the system (3.5). We shall call such a level impenetrable.

Case 7. The level $I = I_*$ is nonresonant, $I_* \neq I_0$ and (1.8) holds. The behavior of solutions in the neighborhood U_{ε^s} of such levels is described, in accordance with (2.8), by the following truncated system with the accuracy of up to terms of order $O(\varepsilon)$:

$$u^{\cdot} = \varepsilon^{1-s}(B_0 + R_N(I_*, \theta, \varphi)), \ \theta^{\cdot} = \omega_* + \varepsilon^{1-s} b_m u^{m+1} \tag{3.8}$$

This case is analogous to case 3, i.e. the level $I = I_*$ is penetrable.

Case 8. The level $I = I_*$ is nonresonant, $I_* = I_0$ and (1.8) holds. Here, as in case 4, a ring of width of order ε^s in I can be shown, into which the trajectories of the mapping converge when $\varepsilon B_1(I_*) < 0$.

4. Let us bring into (3.2) the detuning factor γ defining the deviation of the resonant level from the level $I = I_0$, and consider the passage when γ varies from the exact resonance, to the nonresonant case. To do this we consider a system on a cylinder

$$du/d\tau = A_{0*}(\psi) + \mu(\sigma(\psi) u + \gamma), \ d\psi/d\tau = bu + \mu b_1 u^2 \tag{4.1}$$

obtained from (3.2) by the time change $\tau = \mu t$ and when $B_0(I_{pq}) = (dB_0(I_0)/dI)(I_{pq} - I_0) = \mu\gamma$. We consider, together with (4.1), the conservative system (3.4)

$$du/d\tau = A_{0*}(\psi), \ d\psi/d\tau = bu \tag{4.2}$$

The separatrices of the saddles of system (4.2) enveloping the phase cylinder $\{\psi \bmod 2\pi, u\}$ shall be called the outer separatrices. Let us establish the relative distribution of the outer separatrices, with terms $O(\mu)$ taken into account. We use the formula for determining the distance $\Delta_\mu = \mu\Delta_1 + O(\mu^2)$ separating the corresponding separatrices of the system (4.1) /11/. We take (4.2) as the unperturbed system, and make in (4.1) the substitution

$$\psi = \xi + \psi_0 - \mu\gamma / A_{0*}'(\psi_0), \ u = \eta \tag{4.3}$$

(ψ_0 is the coordinate of the saddle of (4.2)). As a result, (4.1) is transformed, with the accuracy of up to terms of order $O(\mu^2)$, to the form

$$\begin{gathered} d\eta/d\tau = A_{0*}(\xi + \varphi_0) + \mu[\gamma + \sigma(\xi + \psi_0)\eta - \gamma A_{0*}'(\xi + \psi_0)/A_{0*}'(\psi_0)] \\ d\xi/d\tau = b\eta + \mu b_1 \eta^2 \end{gathered} \tag{4.4}$$

Clearly, the right-hand sides of the system (4.4) vanish when $\xi = \eta = 0$, therefore we have, in accordance with /11/,

$$\Delta_1 = -\int_{-\infty}^{\infty} b_1 \eta^2 \frac{d\eta}{d\tau} d\tau + \int_{-\infty}^{\infty} \left[\sigma(\xi + \psi_0)\eta + \gamma - \gamma \frac{A_{0*}'(\xi + \psi_0)}{A_{0*}'(\psi_0)}\right] \frac{d\xi}{d\tau} d\tau \tag{4.5}$$

where $\xi(\tau), \eta(\tau)$ is a solution of the system (4.2) on the separatrix. The integral of the system (4.2) yields the following relation connecting η and ξ:

$$\eta = \pm\left[\frac{2}{b}(V(\xi, \psi_0) - V(0, \psi_0)\right]^{1/2}, \quad V(\xi, \psi_0) = \int A_{0*}(\xi + \psi_0)\, d\xi \tag{4.6}$$

Substituting (4.6) into (4.5), we find

$$\Delta_1^{\pm} = \frac{2\pi\gamma}{p} \pm \int_0^{2\pi/p} \sigma(\xi + \psi_0)\left[\frac{2}{b}(V(\xi, \psi_0) - V(0, \psi_0))\right]^{1/2} d\xi \tag{4.7}$$

Assertion 2. When the function $\sigma(\psi)$ is sign definite, the resonance is exact and $\gamma = 0$, the outer sepratrices are split, i.e. $\Delta_\mu \neq 0$. Indeed, the lower bound of the integral in (4.7) is given by

$$\sigma_{\min} \int_0^{2\pi/p} \left[\frac{2}{b}(V(\xi,\psi_0) - V(0,\psi_0))\right]^{1/2} d\xi$$

which is proportional to the area bounded by the curve (4.6) and the ξ-axis. When $\gamma = 0$, we can write for /4.1/ the generating Poincaré– Pontriagin equation /10/ to prove the following assertion.

Assertion 3. When the function $\sigma(\psi)$ is sign definite, then the system (4.1) with $\gamma = 0$ has no limit cycles enveloping, or not enveloping the phase cylinder.

When the function $\sigma(\psi)$ is sign definite, the conditions $\Delta^{\pm} = 0$ yield the bifurcation values of the detuning factor $\gamma = \gamma^{\pm}$ corresponding to the presence in the system (4.1) (within the accuracy used) or a separatrix passing from a saddle to a saddle, and determined by equating the right-hand side of (4.7) to zero. The plus sign corresponds to the region $\eta > 0$, and minus to $\eta < 0$. On varying γ from $\gamma^{\pm}$, a limit cycle is generated in the system (4.1) enveloping the phase cylinder, and from this follows:

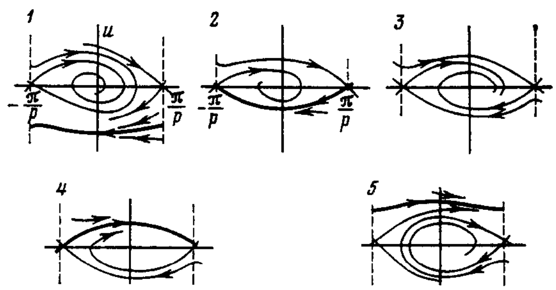

Fig.2

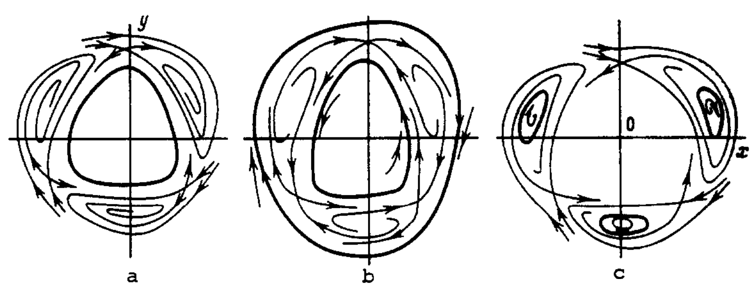

Fig.3

Assertion 4. Let $\sigma(\psi, I_{pq})$ be a sign definite function, and let us assume for definiteness that $\varepsilon\sigma < 0$. Then $\gamma_{pq} = |\gamma^{\pm}| + O(\mu)$ can be found such that 1) when $\gamma > \gamma_{pq}$, then the system (4.1) has a stable limit cycle enveloping the phase cylinder $\{\psi \bmod 2\pi, u\}$; 2) when $\gamma = \gamma_{pq}$, the limit cycle becomes "imbedded" in the separatrix contour Γ_p^+ consisting of p saddles and outer separatrices passing from saddle to saddle, with the remaining unstable separatrices of the saddles tending, with $t \to \infty$, to stable foci; 3) when $-\gamma_{pq} < \gamma < \gamma_{pq}$ no limit cycles enveloping the phase cylinder exist; 4) when $\gamma = -\gamma_{pq}$ a contour Γ_p^- forms composed of p saddles and outer separatrices and differing from Γ_p^+ in the direction of the passage around the phase cylinder and its position on it; 5) when $\gamma < -\gamma_{pq}$, a stable limit cycle exists enveloping the phase cylinder.

Fig.2 illustrating the above assertion shows the phase patterns of the system (4.1) in the strip $\{\psi \bmod (2\pi/p), u\}$ for various values of detuning γ, and Fig.3a depicts the phase pattern corresponding to Fig.2 (1) in initial variables x and y for $p = 3$ and $q = 1$.

Let us turn our attention to the case of the sign variable function $\sigma(\psi)$. The system (4.1) is close to the Hamiltonian system (4.2), with the Hamiltonian

$$H_*(\psi, u) = bu^2/2 - V(\psi)$$

We introduce, in the region of rotational motions $(H_* = h,\ h > V(\psi_0))$ of the system (4.2), the action J-angle α variables, write the system (4.1) in terms of these variables, and average it over the rapid variable α. This yields the equation

$$\bar{J}^{\cdot} = \mu P_0(\bar{J}), \quad P_0(\bar{J}) = \frac{1}{2\pi}\int_0^{2\pi} [(\sigma(\psi(\bar{J},\alpha))\,u(\bar{J},\alpha) + \gamma)\,\psi_\alpha' - b_1 u^2 u_\alpha']\,d\alpha = \pm\frac{\gamma}{p} + \frac{b}{2\pi\Omega}\int_0^{2\pi}\sigma(\psi(\bar{J},\alpha))\,u^2(\bar{J},\alpha)\,d\alpha, \quad \Omega = d\alpha/d\tau \tag{4.8}$$

The plus sign corresponds to the upper half-cylinder, and the minus sign to the lower. The equation $P_0(\bar{J}) = 0$ is a generating equation; simple real roots $\bar{J}_k$ of this equation determine the coarse limit cycles in the system (4.1) lying near the levels $H_* = h(\bar{J}_k)$. From (4.8) follows that when $\gamma = 0$ the number of the limit cycles of (4.1) on the upper half-cylinder coinciding with the number of limit cyles on the lower half-cylinder (Fig.3b). Further, $\gamma_{pq}^* > 0$ exists that when $|\gamma| > \gamma_{pq}^*$, then the system (4.1) has a single cycle only. In concrete cases the formula (4.8) enables us to solve the problem of zeros of the generating function $P_0(\bar{J}, \gamma)$ depending on the values of the parameter γ, and thus obtain the modification of the phase pattern of (4.1) in the region of "rotational motions". We note that in the region of "oscillatory motions" of (4.1) the generating equation is independent of γ.

5. In conclusion we give an example of the system

$$x^{\cdot} = y,\ y^{\cdot} = -x + x^3 + \varepsilon\,[(\delta + x\cos t)\,y + \beta\sin t] \tag{5.1}$$

for which the autonomous system (1.4) has no limit cycles. In spite of this we find that in the neighborhood U_μ of the levels $I = I_{pq}$ (p is odd and $q = 1$) the system (5.1) has, for certain definite values of the parameters δ and β, two-frequency resonant modes corresponding to the limit cycles not enveloping the phase cylinder of the system (3.2) (see Fig.3c where $p = 3$).

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